On the convex hull of k-additive 0-1 capacities and its application to model identification in decision making

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- In this paper, we extend this mechanism to 3-additive capacities, thanks to results we establish on the convex hull of k-additive 0-1 capacities.

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- $\mathcal{M}_k(n)$: set of (at most) k-additive capacities (closed convex polytope of dimension $d(n,k) := \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} - 1$)

The identification problem

- Models of preference based on capacities (using the Choquet integral, etc.) are identified through an optimization problem whose variables are the coefficients $v(S)$ or equivalently $m^v(S)$, $S \subseteq N$, with $n2^{n-1}$ monotonicity constraints.

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- When k-additive capacities are used, the number of variables becomes polynomial: $d(n, k)$. However, the number of constraints is still $n2^{n-1}$.

n	5	10	15	20	25
Number of monotonicity constraints	80	5 120	245 760	10 485 760	419 430 400

The identification problem

- Alternative way: as $\mathcal{M}_2(n)$ is a polytope, use its vertices to represent v as a convex combination:

$$v = \sum_{i \in N} w_i \, u_{\{i\}} + \sum_{\{i,j\} \subseteq N} w_{i,j} \, u_{\{i,j\}} + \sum_{\{i,j\} \subseteq N} \overline{w_{i,j}} \, \overline{u_{\{i,j\}}}$$

where $u_{\{i\}}$, $u_{\{i,j\}}$ are unanimity games, and

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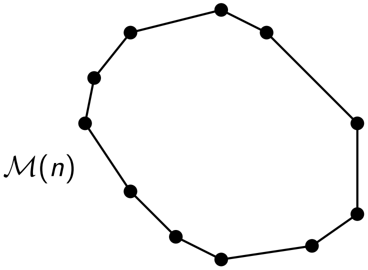
- Then the variables are the coefficients $w_i, w_{i,j}, \overline{w_{i,j}}$, and the constraints are:

$$w_i \geqslant 0, \quad w_{i,j}, \overline{w_{i,j}} \geqslant 0 \quad (\forall i,j)$$

$$\sum_{i\in N} w_i + \sum_{\{i,j\}\subseteq N} w_{i,j} + \sum_{\{i,j\}\subseteq N} \overline{w_{i,j}} = 1.$$

	Möbius representation	vertices representation
Number of unknowns	$\frac{n(n+1)}{2}$	n^2
Number of monotonicity conditions	$n\,2^{n-1}$	n^2
Number of normalization conditions	2	1

The situation beyond $k = 2$



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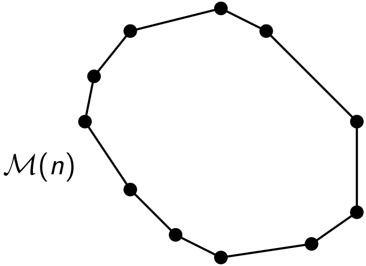

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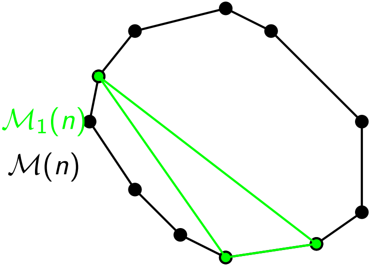

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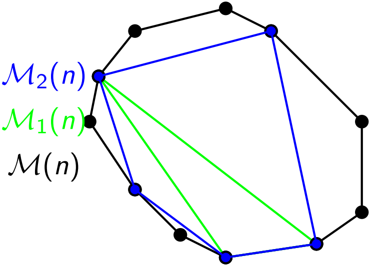

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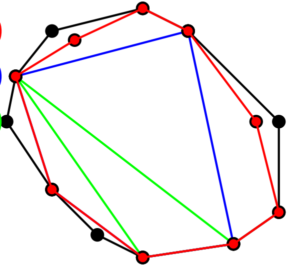

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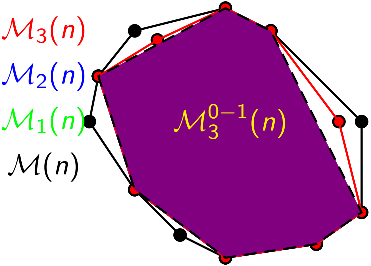
- Idea: use only 0-1 3-additive capacities: $\mathcal{M}_3^{0-1}(n)$ and take the convex hull $\mathrm{conv}(\mathcal{M}_3^{0-1}(n))$



Volume of $\mathcal{M}(n)$

As $\mathcal{M}(n)$ is an *order polytope*, its volume is given by

$$V(\mathcal{M}(n)) = \frac{e(2^N)}{(2^n - 2)!}$$

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n	$e(2^N)$
1	1
2	2
3	48
4	1680384
5	14807804035657359360
6	141377911697227887117195970316200795630205476957716480

Volume of $\mathcal{M}(n)$

n	$V(\mathcal{M}(n))$
1	1
2	1
3	0.0666667
4	0.0000192753
5	0.0000000000000558252
6	0.00000000000000000000000000000000449247

Vertices of $\mathcal{M}(n)$

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n	$M(n) - 2$
1	1
2	4
3	18
4	166
5	7579
6	7828352
7	2414682040996
8	56130437228687557907786

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The key result to obtain them is: Take an antichain $\mathcal{A} = \{A_1, \ldots, A_\ell\}$, its corresponding simple game has Möbius transform:

$$m^v(S) = \sum_{\substack{I \subseteq \{1,\ldots,\ell\} \\ I \neq \varnothing \\ \bigcup_{i\in I} A_i = S}} (-1)^{|I|+1} \qquad (S \subseteq N, S \neq \varnothing)$$

with the convention $\sum_{\varnothing} = 0$.

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Lemma

Consider an antichain $\mathcal{A}$ in $(2^N \setminus \{\varnothing, N\}, \subseteq)$ such that $|\bigcup \mathcal{A}| \leqslant k$. Then the capacity generated by $\mathcal{A}$ belongs to $\mathcal{M}_k^{0-1}(n)$.

Characterization of vertices of $\mathrm{conv}(\mathcal{M}_3^{0-1}(n))$

Theorem

Let $\mathcal{A}$ be an antichain on $(2^N \setminus \{\varnothing, N\}, \subseteq)$, with support $|\bigcup \mathcal{A}| > 3$, and denote by $v_{\mathcal{A}}$ the corresponding 0-1 capacity. Then $v_{\mathcal{A}} \in \mathcal{M}_3^{0-1}(n)$ iff

1. *No element of $\mathcal{A}$ has a cardinality larger than 3 and smaller than 2.*
2. *If all elements of $\mathcal{A}$ have cardinality 2, then $\mathcal{A}$ has the form*

$$\mathcal{A} = \{\overline{12}, \overline{34}, \overline{13}\},$$

   i.e., a partition of the support in two blocks and a set formed by an element of each block (support of size 4).

3. *If $\mathcal{A}$ has an element of cardinality 3, say, $\overline{123}$, then $\mathcal{A}$ has the form*

$$\mathcal{A} = \{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}$$

   (support of size 4).

4. *The size of the support of $\mathcal{A}$ is at most 4.*

Characterization of vertices of $\mathrm{conv}(\mathcal{M}_3^{0-1}(n))$

antichain	vertex	number
$\{\{i\}\}$	$u_{\overline{i}}$	n
$\{\{i,j\}\}$	$u_{\overline{ij}}$	$\binom{n}{2}$
$\{\{i\},\{j\}\}$	$u_{\overline{i}} + u_{\overline{j}} - u_{\overline{ij}} =: \overline{u}_{ij}$	$\binom{n}{2}$
$\{\{i\},\{j\},\{\ell\}\}$	$u_{\overline{i}} + u_{\overline{j}} + u_{\overline{\ell}} - u_{\overline{ij}} - u_{\overline{i\ell}} - u_{\overline{j\ell}} + u_{\overline{ij\ell}} =: \overline{u}_{ij\ell}$	$\binom{n}{3}$
$\{\{i\},\{j,\ell\}\}$	$u_{\overline{i}} + u_{\overline{j\ell}} - u_{\overline{ij\ell}} =: \overline{u}_{i,j\ell}$	$n\binom{n-1}{2}$
$\{\{i,j\},\{j,\ell\}\}$	$u_{\overline{ij}} + u_{\overline{j\ell}} - u_{\overline{ij\ell}} =: \overline{u}_{ij,j\ell}$	$(n-2)\binom{n}{2}$
$\{\{i,j\},\{j,\ell\},\{i,\ell\}\}$	$u_{\overline{ij}} + u_{\overline{j\ell}} + u_{\overline{i\ell}} - 2u_{\overline{ij\ell}} =: \overline{u}_{ij,j\ell,i\ell}$	$\binom{n}{3}$
$\{\{i,j,\ell\}\}$	$u_{\overline{ij\ell}}$	$\binom{n}{3}$
$\{\{i,j\},\{s,t\},\{i,s\}\}$	$u_{\overline{ij}} + u_{\overline{st}} + u_{\overline{is}} - u_{\overline{ijs}} - u_{\overline{ist}} =: \overline{u}_{ij,st,is}$	$2\binom{n}{2}\binom{n-2}{2}$
$\{\{i,j,s\},\{i,t\},\{j,t\},\{s,t\}\}$	$u_{\overline{ijs}} + u_{\overline{it}} + u_{\overline{jt}} + u_{\overline{st}} - u_{\overline{ijt}} - u_{\overline{ist}} - u_{\overline{jst}} =: \overline{u}_{ijs,t}$	$(n-3)\binom{n}{3}$

Total number of vertices $= n\Big(n + \frac{1}{6}(n-1)(n-2)(4n-3)\Big)$ (this is in $O(n^4)$).

Facets

The facets of $\mathcal{M}_k(n)$ are known. They correspond to the $n2^{n-1}$ monotonicity inequalities:

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By contrast, the facets of $\mathrm{conv}(\mathcal{M}_k^{0-1}(n))$ are unknown. However:

Theorem

Any facet of $\mathcal{M}_k(n)$ is a facet of $\mathrm{conv}(\mathcal{M}_k^{0-1}(n))$.

The case $n = 4$, $k = 3$

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- By using PORTA[1], it is possible to find the facets of $\mathrm{conv}(\mathcal{M}_3^{0-1}(4))$, as well as all vertices of $\mathcal{M}_3(4)$. We find:

	$\mathcal{M}_3(4)$		$\mathrm{conv}(\mathcal{M}_3^{0-1}(4))$
vertices	303	$\supseteq$	68
facets	32	$\subseteq$	222

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- By using VINCI[2] and LRS[3], it is possible to compute the volumes of $\mathcal{M}_3(4)$ and $\mathrm{conv}(\mathcal{M}_3^{0-1}(4))$ and their ratio. We find:

Volume of $\mathcal{M}_3(4)$	V_1=0.000019927
Volume of $\mathrm{conv}(\mathcal{M}_3^{0-1}(4))$	V_2 =0.000019046
ratio V_2/V_1	0.95581

[1] POlyhedron Representation Transformation Algorithm, by Thomas Christof and Andreas Loebel https://porta.zib.de/

[2] by Benno Büeler and Andreas Enge https://www.multiprecision.org/vinci/

[3] by David Avis http://cgm.cs.mcgill.ca/~avis/C/lrslib/

Back to the identification problem

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  - the Choquet integral is linear w.r.t. the capacity:

$$\int f \, \mathrm{d}(v + \alpha v') = \int f \, \mathrm{d}v + \alpha \int f \, \mathrm{d}v'$$

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- The previous results permit to use the set 3-additive capacities in $\mathrm{conv}(\mathcal{M}_3^{0-1}(n))$ in modelling preferences, inducing an optimization problem in $O(n^4)$, both for the number of variables and the number of constraints
- The loss of generality, i.e., the volume of $\mathcal{M}_3(n) \setminus \mathrm{conv}(\mathcal{M}_3^{0-1}(n))$, seems to be small (relative volume about 5% for $n = 4$)
- Case of the Choquet integral: it is easy to get the expression of the Choquet integral for each capacity in $\mathcal{M}_3^{0-1}(n)$, remembering that
  - the Choquet integral is linear w.r.t. the capacity:

$$\int f \,\mathrm{d}(v + \alpha v') = \int f \,\mathrm{d}v + \alpha \int f \,\mathrm{d}v'$$

  - the Choquet integral w.r.t. a unanimity game is given by

$$\int f \,\mathrm{d}u_S = \min_{x \in S} f(x)$$

antichain	Choquet integral
$\{\{i\}\}$	$C_1(x) = x_i$
$\{\{i,j\}\}$	$C_2(x) = x_i \wedge x_j = \mathrm{OS}^2_1(x_i, x_j)$
$\{\{i\},\{j\}\}$	$C_3(x) = x_i + x_j - x_i \wedge x_j = x_i \vee x_j = \mathrm{OS}^2_2(x_i, x_j)$
$\{\{i\},\{j\},\{\ell\}\}$	$C_4(x) = x_i + x_j + x_\ell - x_i \wedge x_j - x_i \wedge x_\ell - x_j \wedge x_\ell$ $+ x_i \wedge x_j \wedge x_\ell = x_i \vee x_j \vee x_\ell = \mathrm{OS}^3_3(x_i, x_j, x_\ell)$
$\{\{i\},\{j,\ell\}\}$	$C_5(x) = x_i + x_j \wedge x_\ell - x_i \wedge x_j \wedge x_\ell$
$\{\{i,j\},\{j,\ell\}\}$	$C_6(x) = x_i \wedge x_j + x_j \wedge x_\ell - x_i \wedge x_j \wedge x_\ell$
$\{\{i,j\},\{j,\ell\},\{i,\ell\}\}$	$C_7(x) = x_i \wedge x_j + x_j \wedge x_\ell + x_j \wedge x_\ell - 2\, x_i \wedge x_j \wedge x_\ell$ $= \mathrm{OS}^3_2(x_i, x_j, x_\ell)$
$\{\{i,j,\ell\}\}$	$C_8(x) = x_i \wedge x_j \wedge x_\ell = \mathrm{OS}^3_1(x_i, x_j, x_\ell)$
$\{\{i,j\},\{s,t\},\{i,s\}\}$	$C_9(x) = x_i \wedge x_j + x_s \wedge x_t + x_i \wedge x_s - x_i \wedge x_j \wedge x_s$ $- x_i \wedge x_s \wedge x_t$
$\{\{i,j,s\},\{i,t\},\{j,t\},\{s,t\}\}$	$C_{10}(x) = x_i \wedge x_j \wedge x_s + x_i \wedge x_t + x_j \wedge x_t + x_s \wedge x_t$ $- x_i \wedge x_j \wedge x_t - x_i \wedge x_s \wedge x_t - x_j \wedge x_s \wedge x_t$