Recent results on Fuzzy Measures

Hierarchical and Antibuoyant Fuzzy Measures

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M. Grabisch visit
March, 2023

Recent results on FMs

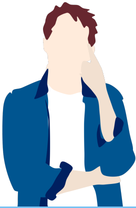
- Hierarchies and Möbius representation

    1. G. Beliakov, M. Gagolewski and S. James. Hierarchical data fusion processes involving the Möbius representation of capacities. Fuzzy Sets and Systems, 433: 1-21, 2022

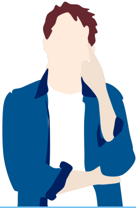
- Antibuoyancy

    2 G. Beliakov and S. James. Choquet integral optimisation with constraints and the buoyancy property for fuzzy measures. Information Sciences. 578: 22–36, 2021

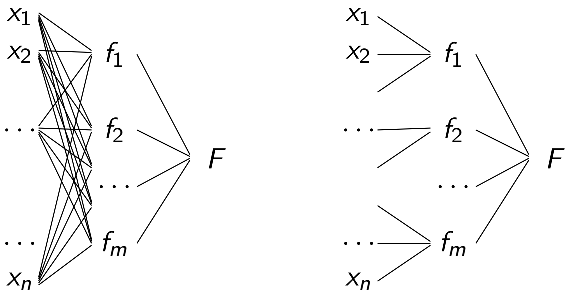
    3 G. Beliakov and S. James. Choquet integral based measures of economic welfare and species diversity. International Journal of Intelligent Systems. 37 (4): 2849 - 2867, 2022



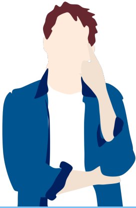
Hierarchical FMs



Hierarchical FMs - background



The relationship between certain hierarchical aggregation structures and fuzzy measures has been investigated by Mesiar et al. and Sugeno et al. in the 1990s.

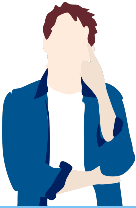


Hierarchical FMs

Proposition

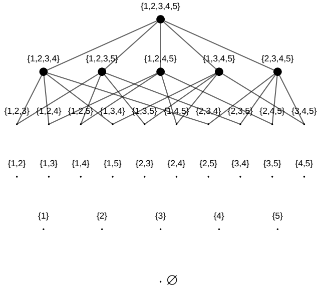
Consider an input vector $\mathbf{x} = (x_1, \ldots, x_n)$ *and a covering* $\{A_1, \ldots, A_m\}$, *i.e., one for which* $A_i \neq \emptyset$ *for all* i *and* $\bigcup_{i=1}^{m} A_i = N$. *For any weighted arithmetic mean of Choquet integrals* $C_{\mu^{(1)}}(\mathbf{x}_{A_1})$, $C_{\mu^{(2)}}(\mathbf{x}_{A_2})$, $\ldots$, $C_{\mu^{(m)}}(\mathbf{x}_{A_m})$ *there exists an equivalent single Choquet integral* $C_\mu(\mathbf{x})$.

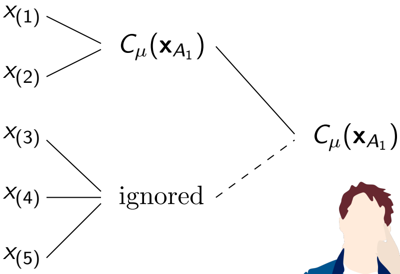
$\rightarrow$ some complex Choquet integrals can be represented as a WAM of Choquet integrals in order to reduce variables and fitting constraints.

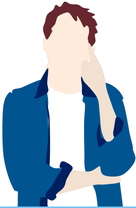


Hierarchical FMs

Some simplified FMs can be interpreted in the hierarchical framework, e.g. k-intolerant (ordered hierarchy)

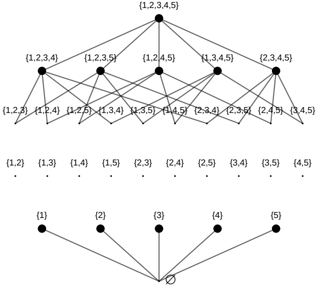


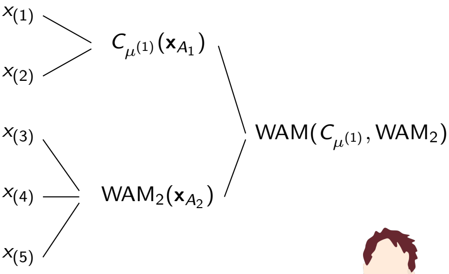


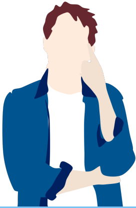


Hierarchical FMs

k-lower interactive

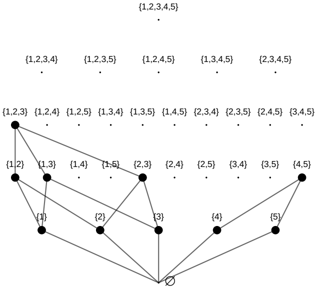


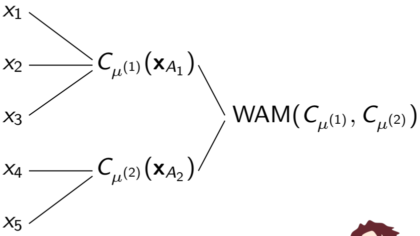


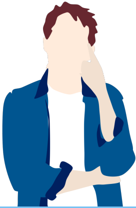


Hierarchical FMs

Partition

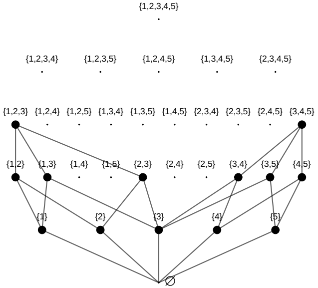


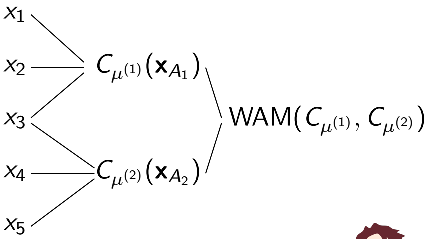


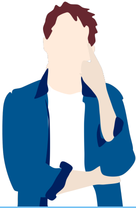


Hierarchical FMs

Overlapping Hierarchy



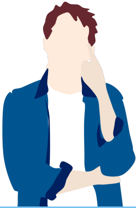




Hierarchical FMs

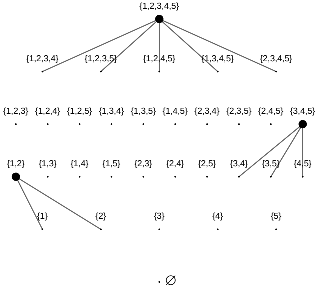
Unidentified parameters (var) and constraints (constr) required for hierarchical fuzzy measures based on partitions with disjoint subsets of at most cardinality k

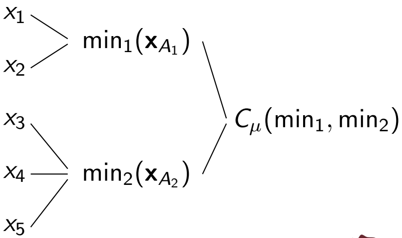
	$n=5$		$n=10$		$n=100$	
k	var	constr	var	constr	var	constr
1	5	5	10	10	100	100
2	7	9	15	20	150	200
3	10	16	22	37	232	397
4	16	33	33	68	375	800
5	31	80	62	160	620	1600
10			1023	5120	10230	51200

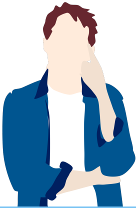


Hierarchical FMs

Other Hierarchies







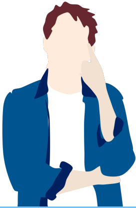
Hierarchical FMs - Fitting comparison

$n = 5$, 100 experiments, fitting to a vertices of a randomly generated FM (linear extension method), L1 error

operator	v	mon	av L1 error
WAM	5	-	0.4015 (0.073)
OWA	5	-	0.3370 (0.053)
WAM+OWA	10	-	0.2595 (0.039)
2-add	15	75	0.2802 (0.079)
1-lo int	10	25	0.2629 (0.051)
2-lo int	20	55	0.1233 (0.035)

Random subsets		
v	mon	av L1 error
8	11.2	0.3336 (0.067)
10	20.1	0.2991 (0.063)
15	41.3	0.2231 (0.053)
20	57.7	0.1542 (0.043)
25	68.0	0.0935 (0.036)

* Preliminary Experiments, not published

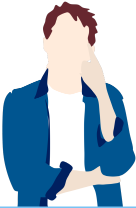


Random Subsets - Potential use

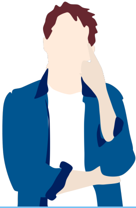
Random Forest / Genetic Algorithm - type processes, e.g.

First, learn multiple FMs with the non-zero subsets chosen randomly (other than 1..n)

- Learn the best FM from the data with these as a basis
- Randomly Combine or take a WAM based on error to obtain an overall FM



Antibuoyant FMs



Anbtibuoyant FMs - background

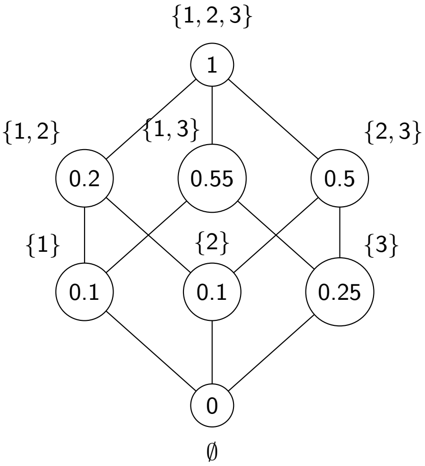
Buoyancy/Antibuoyancy is terminolgoy adopted for OWA (Yager 1993)

$\rightarrow$ Buoyant means weights are non-increasing (largest weight applied to the largest input)

When antibuoyant OWA are used for *Welfare indices* (e.g. Aristondo et al. 2013), the corresponding measures satisfy the **Pigou Dalton** principle – i.e., if the inputs are incomes, then *any proportional transfer of wealth from richer to poorer should increase overall welfare*

Anbtibuoyant FMs - background

In [2], we extended this definition to FMs, e.g.



$\mathbf{w} = (0.25, 0.25, 0.5)$

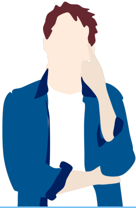
$\mathbf{w} = (0.25, 0.3, 0.45)$

$\mathbf{w} = (0.1, 0.4, 0.5)$

$\mathbf{w} = (0.1, 0.1, 0.8)$

$\mathbf{w} = (0.1, 0.45, 0.45)$

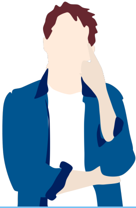
$\mathbf{w} = (0.1, 0.1, 0.8)$



Antibuoyant FMs

This extension has a few nice implications

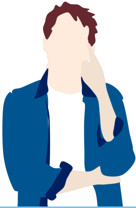
- As far as we are aware, this is the first non-symmetric function to be proposed satisfying the Pigou Dalton principle
  - The idea of non-symmetric functions satisfying PD had mainly been approached using concavity – increases to smaller affect output more than decreases to larger input – but corresponding supermodular FMs are not necessarily antibuoyant and may violate PD so concavity $\neq$ antibuoyancy
  - The PD idea is important not only in economics but also ecology (species diversity)
- If the objective function is defined by an antibuoyant FM, only need to look on one simplex to find the optimum point



Antibuoyant FMs - Learning

Fitting in general requires a large number of constraints

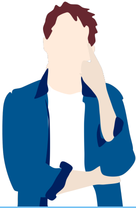
n	variables	monotonicity	antibuoyancy
2	2	4	2
3	6	12	12
4	14	32	48
5	30	80	160
6	62	192	480
7	126	448	1344
8	254	1024	3584
9	510	2304	9216
10	1022	5120	23040



Antibuoyant FMs - Learning

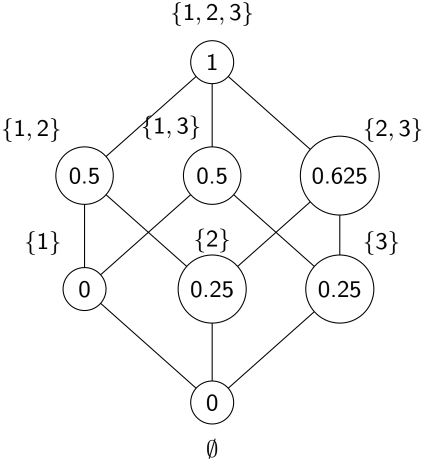
We have proposed an approximate fitting method using vertices of the antibuoyant polytope. There are more vertices than there are for the general FMs polytope, however we defined one type that is p-symmetric and can be generated algorithmically.
e.g. a reference set $A = \{1, 2, 3\}$ for $n = 5$

		$\lvert B \cap A \rvert$			
		0	1	2	3
$\lvert B \cap A' \rvert$	0	0	0	0	0
	1	1/16	1/8	1/4	1/2
	2	7/32	3/8	5/8	1

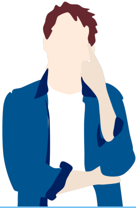


Anbtibuoyant FMs - background

Another example, $A = \{1\}$ and $n = 3$

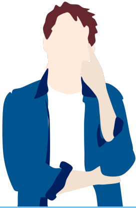


		$\lvert B \cap A \rvert$	
		0	1
$\lvert B \cap A' \rvert$	0	0	0
	1	1/4	1/2
	2	5/8	1



Antibuoyant FMs - Learning

- Such antibuoyant vertices can also be used to generate random antibuoyant FMs
- Generating antibuoyant FMs based on a linear extension method tends toward FMs closer to the minimum – so we also looked at augmenting these with the symmetric and additive FM to test fitting performance.



Thank-you